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Hall Current Effects on Tearing

Modes in Rotating Reverse Field Plasmas

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1. INTRODUCTION

The purpose of the present work is to re-consider a well-known resistive MHD instability by considering the effect of a more general form of Ohm's Law. Specifically, we treat the m=o, finite wavelength tearing mode in cylindrical geometry for a reversed field plasma equilibrium typical of those which are produced in reverse-bias theta pinch experiments[1-5]. The motivation for the present study stems from early experiments which showed a curious and unexplained absence of an expected tearing mode[3]. Reverse bias theta pinch experiments generally produce a closed field configuration with reverse trapped magnetic flux; this occurs due to magnetic line reconnection at the cold ends of the pinch coil where the resistivity is high. The formation of the closed region is also associated with a strong axial contraction as the plasma attempts to find its equilibrium. The result is an elongated (length to width ratio > 10) reversed-field plasma with a bounding closed separatrix, which according to the simplest classical picture should be further susceptable to tearing modes, m=o and $k_z \neq 0$, which would tend to break the column into a number of cells each having reversed trapped flux. The number and size of the cells as predicted by the simple linear theory varies from experiment to experiment since the linear dispersion relation depends sensitively on plasma profiles. The classical tearing mode theory due to Furth, Killeen and Rosenbluth[6] carried out originally in slab geometry yields for cylindrical[7] geometry the prediction that most of the experimentally

produced reversed field theta pinch plasmas should break up into reversed field cells. Experimental results of Eberhagen and Grossmann[3] and more recently by Linford[5] show that the expected tearing mode does not occur and the plasmas in question exist for some tens of linear growth times. Instead of the expected tearing a pronounced rotation of the plasma column is observed; this rotation increases with time until the rotational frequency reaches a critical value at which time the plasma goes catastrophically unstable to an $m=2$ mode. In an attempt to explain the absence of the tearing modes Kalek[7] simulated the plasma behaviour with a linearized initial value MHD code assuming the plasma column to be infinitely long with no longitudinal spatial dependence. Kalek's model originally contained the resistive MHD equations of motion linearized about a rigidly rotating plasma equilibrium; results from the numerical code showed that the inclusion of rotation predicted even larger growth rates than for the static equilibrium case. The addition of the Hall effect into the Ohm's Law yielded results which were not inconsistent with the experimental results, namely the growth rates as calculated were at least an order of magnitude lower than what the FKR theory would predict; this statement requires qualification in that the relevant part of the FKR cylindrical tearing mode dispersion relation is distorted in a manner to suppress the growth rate. The long wavelength ($ka \leq 1$) part of the dispersion relation is affected and a preferred direction of rotation is indicated for a stabilizing influence. The direction of the ion diamagnetic drift

velocity is preferred for stability and rotation in the opposite direction results in enhanced instability. Kalek's numerical results lend strong support to a beneficial interplay between the effects of rotation and Hall current in the suppression of tearing mode growth rates. The calculation described in the present paper attempts to analytically investigate the $m=o$, $k_z \neq 0$ tearing mode in reversed field plasmas and attempts to explain the numerical results of Kalek. Our results are consistent with Kalek's and we find a stabilizing influence of rotation combined with Hall effect; however, we find that the results are very sensitive to profiles, wall position and plasma parameters such as density, temperature and rotation frequency. The specific calculation presented in the present paper deals with plasma profiles which are representative of those which are produced in reversed field theta pinch experiments.

It should be mentioned that the use of Ohm's Law with a Hall term has also been used to model the effect of finite particle effects in treating tearing modes since the Hall term is associated with a coefficient which is proportional to the ion gyration frequency.[8] In such a case overstable modes can be found. In the problem treated here the m=0 mode does not admit overstable solutions; this is not explicitly pointed out in the text but follows from simple energy arguments.

The paper is organized into 14 sections. The governing equations are presented in sections 2 and 3. The governing equations are non-dimensionalized in section 4. Solutions of the

governing equations in the resistive boundary layer are dis-
cussed in sections 5 and 6. Solutions outside the resistive
layer are discussed in section 7 and matching across the
boundary layer is discussed in section 8. The tearing mode
ordering with a small parameter is discussed in section 9 and
10. Sections 11 through 14 discuss the computation of growth
rates for specific equilibria. A final section presents the
conclusions of the computations discussed here.

2. The Model

We consider a straight cylindrical plasma column
whose equilibrium profiles are considered to have
variation only in the radial direction. The plasma is
considered incompressible and has a density profile,
$\rho_0(r)$, that extends to a perfectly conducting wall. The
equilibrium magnetic field, $\underset{\sim}{B}_0(r)$ has both longitudinal
and azimuthal components. Finally, the column is sub-
jected to a rotation with rotational frequency $\Omega(r)$.

To study this system we employ the following
mathematical model:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{v}) = 0 \tag{1}$$

$$\rho \frac{\partial \underset{\sim}{v}}{\partial t} + \rho \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} = -\nabla p + \underset{\sim}{J} \times \underset{\sim}{B} \tag{2}$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla \times \underset{\sim}{E} = 0 \tag{3}$$

$$\underset{\sim}{J} = \frac{1}{\mu_0} \nabla \times \underset{\sim}{B} \tag{4}$$

$$\underset{\sim}{E} = -\underset{\sim}{v} \times \underset{\sim}{B} + \eta \underset{\sim}{J} + \frac{\varepsilon}{\rho} \underset{\sim}{J} \times \underset{\sim}{B} \tag{5}$$

$$\nabla \cdot \underset{\sim}{v} = 0 \tag{6}$$

$$\nabla \cdot \underset{\sim}{B} = 0 \tag{7}$$

where $\underset{\sim}{v}$ is the plasma velocity, $\underset{\sim}{B}$ the magnetic field, p the plasma pressure, $\underset{\sim}{J}$ the current density, $\underset{\sim}{E}$ the electric field and η, ε are resistivity and Hall current parameters respectively which are taken to be constant. The mathematical model specified by Equations (1) - (7) differs from Ideal MHD only through Ohm's Law which includes the effect of finite resistivity and the Hall effect. Our aim in this report is to study the time behavior of small perturbations about an equilibrium specified by $\underset{\sim}{v}_o(r)$, $\underset{\sim}{B}_o(r)$, $\rho_o(r)$ and $p_o(r)$. The perturbed velocity field, magnetic field, density, pressure, current density and electric field are represented by $\underset{\sim}{u}$, $\underset{\sim}{b}$, $\delta\rho$, δp, $\underset{\sim}{j}$ and $\delta\underset{\sim}{E}$ respectively.

The linearized version of equations of (1) - (7) are:

$$\frac{\partial}{\partial t} \delta\rho + \nabla \cdot (\delta\rho\underset{\sim}{v}_o) + \nabla \cdot (\rho_o\underset{\sim}{u}) = 0 \qquad (1a)$$

$$\frac{\partial}{\partial t} \rho_o \underset{\sim}{u} = -\nabla p^* + \underset{\sim}{j}\times\underset{\sim}{B}_o + \underset{\sim}{J}_o\times\underset{\sim}{b} - \rho_o \underset{\sim}{u}\cdot\nabla\underset{\sim}{v}_o \qquad (2a)$$

$$- \rho_o \underset{\sim}{v}_o\cdot\nabla\underset{\sim}{u} - \delta\rho\underset{\sim}{v}_o\cdot\nabla\underset{\sim}{v}_o + \frac{1}{\mu_o}\nabla(\underset{\sim}{B}\cdot\underset{\sim}{b})$$

$$\frac{\partial}{\partial t} \underset{\sim}{b} = -\nabla\times\delta\underset{\sim}{E} \qquad (3a)$$

$$\underset{\sim}{j} = \frac{1}{\mu_o} \nabla\times\underset{\sim}{b} \qquad (4a)$$

$$\delta\underset{\sim}{E} = -\underset{\sim}{u}\times\underset{\sim}{B}_o - \underset{\sim}{v}_o\times\underset{\sim}{b} + \eta\underset{\sim}{j} + \frac{\varepsilon}{\rho_o} \{\underset{\sim}{j}\times\underset{\sim}{B}_o + \underset{\sim}{J}_o\times\underset{\sim}{b} - \frac{\delta\rho}{\rho_o} \underset{\sim}{J}_o \underset{\sim}{B}_o\} \qquad (5a)$$

$$\nabla \cdot \underset{\sim}{u} = 0 \qquad\qquad (6a)$$

$$\nabla \cdot \underset{\sim}{b} = 0 \qquad\qquad (7a)$$

The starting set of equations for the computation can be obtained from system (1a) - (7a) by Fourier transforming the perturbed quantities in the ignorable θ, z directions and time, t. As mentioned, we consider only the mode $m = 0$, k finite where m is the azimuthal wave number and k is the axial wave number. Thus,

$$\underset{\sim}{b} = \underset{\sim}{\tilde{b}}(r) \, \exp(i[\omega t - kz])$$

$$\underset{\sim}{u} = \underset{\sim}{\tilde{u}}(r) \, \exp(i[\omega t - kz])$$

etc.

For ease of notation we will omit superscripts from the perturbed transformed variables.

After transforming equations (1a) - (7a) and rearranging we arrive at the following sixth order system of ordinary differential equations:

$$\frac{d}{dr}(\rho_0 \omega^2 \, \frac{1}{r} \, \frac{d(r\xi_r)}{dr}) - i \, \frac{f}{\mu_0} \, \frac{d}{dr} \, (\frac{1}{r} \, \frac{d(rb_r)}{dr}) = -k^2 \rho_0 (r\Omega^2 \, \frac{\frac{d\rho_0}{dr}}{\rho_0}$$

$$-\omega^2 + 4\Omega^2 + 2r\Omega \, \frac{d\Omega}{dr}) \xi_r + i(\frac{1}{\mu_0} [- \frac{d^2 f}{dr^2} + \frac{1}{r} \, \frac{df}{dr} - k^2 f^2]$$

$$- \frac{2k^2 \Omega}{\omega} \, J_{0,z}) b_r - (\frac{2k^2 f}{\omega\mu_0} \, \Omega + \frac{2B_\theta}{r} \, k^2) \, b_\theta \qquad\qquad (8)$$

$$\frac{\eta}{\mu_o} \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (rb_r) \right) = - f\omega\xi_r + \frac{\epsilon f k}{\mu_o \rho_o} b_\theta$$

$$+ \left(i\omega + \frac{k^2\eta}{\mu_o} + \frac{i\epsilon}{\rho_o} k J_{o,z} \right) b_r \qquad (9)$$

$$\frac{\epsilon f}{\mu_o \rho_o} \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (rb_r) \right) - \frac{\eta k}{\mu_o} \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (rb_\theta) \right)$$

$$= -k \left(i\omega + \frac{\eta k^2}{\mu_o} - \frac{if^2}{\omega\mu_o\rho_o} + \frac{i\epsilon k}{\mu_o} \left[\frac{2B_\theta}{r\rho_o} + \frac{B_\theta}{\rho_o^2} \frac{d\rho_o}{dr} \right] \right) b_\theta$$

$$+ i \left(- \frac{\epsilon}{\rho_o^2} \frac{d\rho_o}{dr} \left[\frac{d}{dr} \frac{f^2}{2\mu_o} + k^2 B_{o,\theta} J_{o,z} \right] \right.$$

$$\left. - k\omega \left[J_{o,z} - \frac{2B_{o,\theta}}{\mu_o r} \right] + 2kf\Omega + krf \frac{d\Omega}{dr} \right) \xi_r$$

$$+ \left(\frac{\epsilon}{\mu_o} \left[r \frac{d}{dr} \left(\frac{1}{r\rho_o} \frac{df}{dr} \right) + \frac{k^2 f}{\rho_o} \right] + kr \frac{d\Omega}{dr} - \frac{kf}{\omega\rho_o} J_{o,z} \right) b_r \qquad (10)$$

where $f = kB_{o,z}$. Futhermore, we have considered the equilib-
rium velocity field to be a rotation, $\underline{v}_o = r\Omega(r)\hat{\theta}$ where $\hat{\theta}$
is a unit vector in the azimuthal direction.

3. The Equilibrium

We consider an equilibrium magnetic field profile with vanishing $B_{O,z}$ on a cylindrical surface, $r = r_o$, in the plasma, i.e., a reversed field, while the component, $B_{O,\theta}$ of the field is taken to be finite on $r = r_o$.

Since the plasma is taken to be incompressible, we may impose an arbitrary density profile. Finally, the equilibrium flow is taken to be a rotation with variation in the radial direction, and no flow in the axial direction.

The above equilibrium may be summarized as:

$$\underline{B}_O(r) = B_{O,\theta}\ \hat{\theta} + B_{O,z}\ \hat{z}\ , \quad B_{O,z}(r_o) = 0$$

$$\underline{v}_O(r) = r\Omega(r)\ \hat{\theta}.$$

$$\rho_O(r) \quad \text{arbitrary}.$$

4. Dimensionalization:

We dimensionalize with respect to the equilibrium values at the neutral surface $r = r_o$ as follows:

$$\overline{\eta} = \frac{\eta}{\omega_o a^2 \mu_o}\ , \quad \overline{\epsilon} = \frac{\epsilon}{\rho_o \omega_o \mu_o} \frac{df}{dr}(r_o)$$

$$\bar{\rho}_0 = \frac{\rho_0(r)}{\rho_0(r_0)} \quad , \quad \bar{f} = \frac{f}{a\frac{df}{dr}(r_0)} \, , \, \frac{d\bar{f}}{dr} = \frac{\frac{df}{dr}}{\frac{df}{dr}(r_0)}$$

$$\overline{\frac{d^2f}{dr^2}}(r_0) = a\frac{\frac{d^2f}{dr^2}(r_0)}{\frac{df}{dr}(r_0)} = \alpha_1 \quad , \quad \tau = \frac{r-r_0}{a} \, , \, \bar{r} = \frac{r}{a}$$

$$D_s = -\frac{a}{\rho_0}\frac{d\rho_0}{dr} \quad , \quad \bar{B}_\theta(r_0) = \frac{\bar{k}\, B_{0,\theta}}{a^2\frac{df}{dr}(r_0)}$$

$$\bar{J}_z = \frac{\bar{k}\, J_{0,z}}{a\frac{df}{dr}(r_0)} \quad , \quad \bar{\gamma} = \frac{i\omega}{\omega_0}$$

$$\bar{\Omega} = \frac{2\Omega\bar{k}}{\omega_0} \quad , \quad \bar{k} = ka \, , \, \frac{d\bar{\Omega}}{dr} = \frac{2a\bar{k}}{\omega_0}\frac{d\Omega}{dr}$$

$$\omega_0 = \frac{a}{(\rho_0\mu_0)^{1/2}}\frac{df}{dr}(r_0) \quad , \quad q_r = \frac{ib_r}{a^2\frac{df}{dr}(r_0)}$$

$$q_\theta = \frac{i\bar{k}b_\theta}{a^2\frac{df}{dr}(r_0)}$$

With these dimensionalizations:

$$\bar{f}(\tau) = \tau + \frac{\alpha_1}{2}\tau^2 + \ldots$$

and,

$$\bar{J}_{0,z} = \alpha_2\tau + \ldots$$

We rewrite equations (8) - (10) in dimensionless
form for a layer centered about r_o that is sufficiently
small to neglect variations in the equilibrium quantities.
This layer will be referred to as the boundary layer,
and the equations governing the behavior of the resistive
layer will be known as the resistive boundary layer equations
or RLE. To leading order in τ and omitting the super-
scripts on the dimensionless quantities, the RLE are:

$$\gamma^2 \frac{d^2\xi_r}{d\tau^2} = -\tau \frac{d^2q_r}{dr^2} + (\Omega^2[1 - \frac{r_o D_s}{4}] + \frac{r_o}{2}\Omega\frac{d\Omega}{dr}] + k^2\gamma^2)\xi_r$$

$$-(\frac{1}{r_o} - k^2\tau - \alpha_1 - \frac{i\Omega\alpha_2}{\gamma})q_r + (\frac{\Omega\tau}{\gamma} - \frac{2iB_\theta}{r_o})q_\theta \qquad (8a)$$

$$\eta \frac{d^2q_r}{d\tau^2} = -\gamma\tau\xi_r + \varepsilon\tau q_\theta + (\gamma + k^2\eta + i\varepsilon\alpha_2\tau)q_r \qquad (9a)$$

$$\eta \frac{d^2q_\theta}{d\tau^2} - \varepsilon\tau \frac{d^2q_r}{d\tau^2} = (\gamma + \eta k^2 + \frac{\tau^2}{\gamma} + i\varepsilon B_\theta[\frac{2}{r} - D_s])q_\theta$$

$$+ (\varepsilon D_s\tau[1 + \alpha_2 B_\theta] + \Omega\tau[1 + \frac{r_o}{2\Omega}\frac{d\Omega}{dr}] + \gamma[\alpha_2\tau - \frac{2iB_\theta}{r_o}])\xi_r$$

$$-(\varepsilon[\alpha_1 - \frac{1}{r_o} + D_s + k^2\tau] - \frac{i\alpha_2}{\gamma}\tau^2 + \frac{r_o}{2}\frac{d\Omega}{dr})q_r \qquad (10a)$$

5. Analysis of the RLE

Scaling:

We wish to study the interesting behavior governed by the RLE in the limit of small resistivity and Hall parameter η and ε. Following the suggestion of reference [9], we adopt the following scaling based on a resistive layer thickness of the order of the skin depth with respect to the diffusion time scale, i.e., $\gamma \tau^2 \sim \eta$ where $\tau \sim \gamma$.

$$\gamma = \phi S \ , \ \tau = \phi x \ , \ \eta = \phi^3 \bar{\eta} \ , \ \Omega = 0(1)$$

$$\varepsilon = \phi \bar{\varepsilon}$$

$$\xi_r = \xi + \ldots, \quad q_r = \phi \psi + \ldots \ ,$$

$$q_\theta = \zeta + \ldots$$

Also denote:

$$P = \frac{r_o}{2\Omega} \frac{d\Omega}{dr} \ , \ \delta = \frac{2B_\theta}{r_o}$$

where the superscripted variables are considered to be $0(1)$ quantities while ϕ is a small scale factor. The scaling of the Hall parameter and angular velocity were chosen so that terms with ε and Ω would neither dominate nor disappear from the scaled equations in the lowest order.

Finally, we again omit the superscripts in writing the scaled equations:

$$S^2 \frac{d^2\xi}{dx^2} + x \frac{d^2\psi}{dx^2} = \Omega^2 (1 - \frac{r_o}{4} D_S + P)\xi + (\frac{\Omega x}{S} - i\delta)\zeta \tag{11}$$

$$\eta \frac{d^2\psi}{dx^2} = -Sx\xi + \epsilon x\gamma + S\psi \tag{12}$$

$$\eta \frac{d^2\zeta}{dx^2} - \epsilon \frac{d^2\psi}{dx^2} = (S + \frac{x^2}{S} + i\epsilon\delta[1 - \frac{r_o D_S}{2}])\zeta$$

$$+ (x\Omega[1 + P] - i\delta S)\xi - P\Omega\psi \tag{13}$$

Making use of (12), equations (11) and (13) can be rewritten:

$$S^2 \frac{d^2\xi}{dx^2} = (\frac{x^2 S}{\eta} + \Omega^2 [1 - \frac{r_o D_S}{4} + P]) \xi + (\frac{\Omega x}{S} - \frac{\epsilon x^2}{\eta}$$

$$- i\delta)\zeta - \frac{xS}{\eta} \psi \tag{11a}$$

$$\eta \frac{d^2\zeta}{dx^2} = (S[1 + \frac{x^2}{S}] + \frac{\epsilon^2}{\eta} x^2 + i\epsilon\delta[1 - \frac{r_o}{2} D_S])\zeta$$

$$+ (x\Omega[1+P] - i\delta S - \frac{\epsilon x^2}{\eta} S)\xi + (\frac{\epsilon xS}{\eta} - P\Omega)\psi . \tag{13a}$$

Equations (11) - (13) show that in the presence of azimuthal components, B_θ, of the equilibrium magnetic field, overstable modes are possible due to the imaginary terms

in the equations. For this analysis we suppress these terms by considering only long wavelength modes with sufficiently small k which results in the elimination of δ and α_2 from equations (11) - (13) to leading order.

6. Asymptotic Behavior of RLE:

The system (11a), (12), (13a) constitutes a sixth order homogeneous linear differential equation with an irregular singularity at $x = \infty$. The system may be rewritten in a standard form for irregular perturbation analysis (see reference [10]):

$$\frac{d\underset{\sim}{u}}{dx} = x \underset{\sim}{A}(x) \underset{\sim}{u}$$

which has a set of solutions of the form:

$$\underset{\sim}{u} = \underset{\sim}{c} \, e^{\underset{\sim}{\lambda} \frac{x^2}{2} + \dots} x^{\underset{\sim}{D}} \, \underset{\sim}{\hat{z}}(x)$$

where $\underset{\sim}{c}$ is a constant matrix and $\underset{\sim}{\lambda}$, $\underset{\sim}{D}$ are diagonal matrices.

The elements of $\underset{\sim}{\lambda}$ determine the exponential nature of the asymptotic solution to the inner equations. These elements, λ_i, are the eigenvalues of the matrix,

$$\underset{\sim}{A}(\infty) = \begin{vmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{\eta Q} & -\frac{\epsilon}{\eta Q^2} & 0 & 0 & 0 & 0 \\ -\frac{\epsilon Q}{\eta} & (\frac{1}{\eta Q} + \frac{\epsilon^2}{\eta}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Thus,

$$\lambda_{1,2} = 0,0$$

$$\lambda_{3,4} = \frac{1}{(\eta S)^{1/2}} \left(\frac{r}{2} \pm \left(1 + \frac{r^2}{4}\right)^{1/2}\right)$$

$$\lambda_{5,6} = -\frac{1}{(\eta S)^{1/2}} \left(\frac{r}{2} \pm \left(1 + \frac{r^2}{4}\right)^{1/2}\right)$$

where $r^2 = \frac{\epsilon^2 s}{\eta}$.

At this point it is not clear whether λ_k, $k = 3,4,5,6$ have pure or oscillating exponential growth or decay, or whether they have pure oscillatory behavior as $x \to \infty$. In fact, we can show that only if,

$$S = \frac{\eta}{(\alpha^2+1)\epsilon^2} \left(-1 \pm \frac{i}{\alpha}\right)$$

for α an arbitrary real number, can λ_k be pure imaginary in which case

$$\lambda_k = \pm \frac{\epsilon}{\eta} i .$$

Thus, all these solutions either decay or grow exponentially as $x \to \infty$ in the class of solutions with positive growth rates, i.e., $\mathrm{Re}\, S > 0$,

Finally we consider the two algebraic equations corresponding to λ_1 and λ_2. To determine the asymptotic behavior of these solutions as $x \to \infty$, we consider the

dominant terms from equations (11a) and (13a), as
$x \rightarrow \infty$:

$$\frac{S\xi}{\eta} - \frac{\varepsilon\gamma}{\eta} \simeq \frac{S\psi}{x\eta}$$

$$(\frac{\Omega}{x} - \frac{\varepsilon S}{\eta})\xi + (\frac{1}{S} + \frac{\varepsilon^2}{\eta})\gamma \simeq -\frac{\varepsilon S}{x\eta} \psi$$

Solving to leading order in x for ξ and γ yields:

$$\xi \simeq \frac{\psi}{x} , \tag{14}$$

$$\gamma \simeq -\frac{S\Omega}{x^2} \psi . \tag{15}$$

Thus,

$$\gamma \simeq -\frac{S\Omega}{x} \xi . \tag{16}$$

Also from (12),

$$\frac{d^2\psi}{dx^2} \simeq \frac{d^2(x\xi)}{dx^2} - \varepsilon \frac{d^2}{dx^2} (\frac{x\gamma}{S}) \tag{17}$$

Replacing (14) and (15) in (11) and noting that

$$x \frac{d^2}{dx^2} (x\xi) = \frac{d}{dx} (x^2 \frac{d\xi}{dx})$$

yields, to leading order as $x \to \infty$,

$$\frac{d}{dx} (x^2 \frac{d\xi}{dx}) = - \Omega^2 (\frac{r_o}{4} D_S - P) \xi \qquad (18)$$

which has the solutions,

$$\xi \sim x^h \; , \; x^{-1-h}$$

where $h = - \frac{1}{2} + \frac{1}{2}(1 - \Omega^2 (\frac{r_o D_S}{4} - P))^{1/2}$ in order to arrive at (16) we made use of the fact that in view of (14), $\gamma \sim x^{h-1} \, , \, x^{-2-h}$.

In summary, the asymptotic solutions to system (11)-(13) take the form, as $x \to \infty$,

$$\xi \sim \exp \{\frac{1}{(\eta S)^{1/2}} (\frac{r}{2} \pm (1 + \frac{r^2}{4})^{1/2}) \frac{x^2}{2}\}$$

$$\sim \exp \{\frac{-1}{(\eta S)^{1/2}} (\frac{r}{2} \pm (1 + \frac{r^2}{4})^{1/2}) \frac{x^2}{2}\}$$

$$\sim \quad x^h \, , \, x^{-h-1}$$

where $r = \frac{\varepsilon^2 S}{\eta}$, $h = -\frac{1}{2} + \frac{1}{2} (1 - \Omega^2 (\frac{r_o}{4} D_S - P))^{1/2}$.

7. <u>Solutions Outside the Resistive Layer</u>:

The behavior of the plasma outside a narrow region surrounding the neutral surface is governed to leading order by ideal MHD, i.e., ϵ and η can be taken to be zero. In terms of the same dimensionalizations as the resistive layer equations, the outer equations reduce, near marginal stability, i.e. $\gamma \simeq 0$, to the following second order differential equation:

$$\frac{d}{dr}(\frac{f^2}{r} \frac{d}{dr}(r\xi_r))-f^2k^2\xi_r = -\frac{\rho_o}{4} r\Omega^2 (D_S-\frac{2}{\Omega} \frac{d\Omega}{dr})\xi_r \tag{19}$$

Near r_o equations (11) - (13) reduce asymptotically to:

$$\frac{d}{d\tau} (\tau^2 \frac{d\xi_r}{d\tau}) = -\Omega^2 (\frac{r_o D_S}{4} - P)\xi_r \tag{20}$$

which has solutions,

$$\xi = \tau^h , \quad \tau^{-1-h} \tag{21}$$

where $h = -\frac{1}{2} + \frac{1}{2} (1 - \Omega^2 (r_o D_S - P))^{1/2}.$ (21a)

For the purposes of numerical integration, it is convenient to set $W = \dfrac{\xi_r}{\dfrac{d\xi_r}{dx}}$ which transforms (19) to,

$$\frac{dW}{dx} + p(x) W^2 + q(x) W = 1 \tag{22}$$

$$W(0) = W(1) = 0$$

where,

$$p(x) = \frac{1}{x^2} - 2 \frac{df/dx}{xf} + k^2 - \frac{D_s x \bar{\Omega}^2}{4f^2}$$

and,

$$q(x) = - \left(2 \frac{df/dx}{f} + \frac{1}{x}\right)$$

From (22) and the asymptotic expressions $f \approx \tau$, $\dfrac{df}{dx} \approx 1$, we find that the criteria for entering the asymptotic regime governed by (20) is that

$$M = \left| \frac{8\tau}{x_o^2 D_s \bar{\Omega}^2} \right| \ll 1 .$$

In this formula, the value of τ such that $M = .01$ may be taken as a measure of the thickness of the boundary layer, i.e.,

$$\tau_{b.\ell.} = .01 \frac{x_o^2 D_s \bar{\Omega}^2}{8} \tag{23}$$

We note that according to (23) the boundary layer thickens as the plasma rotates faster.

8. Matching Inner and Outer Solutions:

In order to match the outer solutions, the two exponentially growing solutions to the inner equations must be set equal to zero, leaving the two algebraic solutions for matching.

The condition that the algebraic inner solutions match the outer ideal MHD solutions can be expressed as:

$$\lim_{\tau \to 0} \xi_r^{outer}(\tau) \cong \lim_{x \to \infty} \xi^{inner}(x) \tag{24}$$

From (18) and (19) it is clear that the inner solutions can be matched asymptotically to the outer solutions in the sense of (24).

We will be investigating a class of resistive eigenmodes called tearing modes, that are sensitive to the matching conditions between the inner and outer regions. We subdivide the plasma radius into three regions. Region I is the outer region closest to the magnetic axis. Generally, when treating boundary layers, the boundary layer is called the inner region, and the rest is called the outer region. Here, however, we have two outer regions, one of which is in some

sense inside the boundary layer. We can just call it region I. Region II is then the inner, resistive boundary layer, and region III the other outer region.

From Equation (21) we have seen that the solution in Region I near the boundary layer approaches the form:

$$\xi_{r,I} \simeq A_I |\tau|^h + B_I |\tau|^{-1-h} \tag{25}$$

Likewise in Region III,

$$\xi_{r,III} \simeq A_{III} |\tau|^h + B_{III} |\tau|^{-1-h} \tag{26}$$

Similarly in Region II,

$$\xi \simeq \begin{cases} A_{II,L} |x|^h + B_{II,L} |x|^{-1-h} \text{ as } x \to -\infty \\ \\ A_{II,R} |x|^h + B_{II,R} |x|^{-1-h} \text{ as } x \to \infty \end{cases}$$

The matching condition, (24) can be expressed as,

$$\frac{A_I}{B_I} = \left(\frac{1}{\phi}\right)^{2h+1} \frac{A_{II,L}}{B_{II,L}} \tag{27}$$

and,

$$\frac{A_{III}}{B_{III}} = \left(\frac{1}{\phi}\right)^{2h+1} \frac{A_{II,R}}{B_{II,R}} \tag{28}$$

for $2h+1 = (1 - \Omega^2(r_o D_S - P))^{1/2} > 0$ and where ϕ is the small scale factor.

We expect for most cases that $\frac{A_I}{B}$ and $\frac{A_{III}}{B_{III}}$ will be

finite numbers. Thus to obtain matching, the growth rate must be adjusted in the resistive layer so that $A_{II,L} \overset{-}{\approx} 0$. The exceptional case occurs when B_I vanishes. According to Newcomb's stability criterion [11], this is the condition for marginal ideal stability. It is not surprising that a system that is close to being unstable in the ideal theory will have a strongly modified resistive instability.

These matching constants can be calculated numerically from the outer layer equations as the limit

$$\frac{A_I}{B_I} \cdot \frac{A_{III}}{B_{III}} = \lim_{x \to \mp\infty} \left(\mp x^{-4h-2} \frac{\frac{d}{dx}(x^{1+h}\xi)}{\frac{d}{dx}(x^{-h}\xi)} \right) \tag{29}$$

$$= \lim_{x \to \mp\infty} \left(\mp x^{4h-2} \frac{\frac{d}{dx}(x^{h}\psi)}{\frac{d}{dx}(x^{-1-h}\psi)} \right) \ .$$

In terms of W, (29) can be rewritten,

$$\frac{A_{III}}{B_{III}} = \lim_{\tau \to 0} - |\tau|^{-2h-1} \left[\frac{h+1 + \frac{\tau}{W}}{-h + \frac{\tau}{W}} \right]$$

(30a)

$$\frac{A_I}{B_I} = \lim_{\tau \to 0} - |\tau|^{-2h-1} \left[\frac{h+1 - \frac{\tau}{W}}{-h - \frac{\tau}{W}} \right]$$

The quantity matched to the inner layer is,

$$\Delta' = \frac{A_{III}}{B_{III}} - \frac{A_I}{B_I} \ .$$

(30b)

Our objective in this report is to gain some information about the eigenvalue problem by purely analytic means. To do this, we consider the growth rate, S, and $h \tilde{} 0$. It will prove convenient to match the r-component of the perturbed magnetic field at the edge of the resistive layer rather than the plasma displacements, in which case the asymptotic solutions are

$$q_{r,I} \tilde{} B_I + A_I \tau \ , \quad \text{as } \tau \to 0$$

$$q_{r,III} \tilde{} B_{III} + A_{III} \tau \ , \quad \text{as } \tau \to 0 \ .$$

and ,

$$\psi \simeq \begin{cases} B_{II,L} + A_{II,L} \, x & \text{as } x \to -\infty \\ \\ B_{II,R} + A_{II,R} \, x & \text{as } x \to \infty \end{cases}$$

As in (27) and (28) the matching conditions become,

$$\frac{1}{\phi} \frac{A_{II,L}}{B_{II,L}} \quad , \quad \frac{1}{\phi} \frac{A_{II,R}}{B_{II,R}} = 0(1) \quad . \tag{30}$$

We have seen that the behavior of the exponential solutions of the RLE are

$$\xi \sim \exp \left\{ \pm \frac{kx^2}{2(\eta S)^{1/2}} \right\} \quad \text{for } k \text{ constant} \quad ,$$

as $x \to \pm\infty$. Thus as S becomes small, we must consider the length scale to become small as $S^{1/4}$, otherwise the exponential solutions would grow infinitely rapidly making the equations badly posed. That is, there is an inner layer of significant behavior that is even thinner than the resistive skin depth, by another factor of $S^{1/4}$.

We find the interesting behavior within this new scaling by expanding the parameters and variables of equations (11) - (13) in powers of S as follows:

$$\xi = \xi_o + \ldots$$

$$\psi = S^{1/4} \psi_o + S^{7/4} \psi_1 + \ldots$$

$$\zeta = S^{3/2} \gamma_o + \ldots$$

$$x = S^{1/4} \bar{x} , \qquad \varepsilon = \frac{1}{S^{1/2}} \bar{\varepsilon} , \qquad \Omega = S^{3/4} \bar{\Omega}$$

Equations (11) - (13) reduce in the zero-th order to:

$$\eta \frac{d^2 \psi_o}{dx^2} = 0 \tag{31a}$$

$$\frac{d^2 \xi_o}{d\bar{x}^2} = (\frac{\bar{x}^2}{\eta} - \bar{\Omega}^2 [\frac{r_o D_S}{4} -1-P])\xi_o + (\bar{\Omega}\bar{x} - \frac{\bar{\varepsilon}}{\eta} \bar{x}^{-2}) \ \zeta_o - \frac{\bar{x}}{\eta}\psi_o$$

$$\frac{d^2 \zeta_o}{dx^2} = \bar{x}^2 [1+\frac{\bar{\varepsilon}^2}{\eta}] \zeta_o + (\bar{x}\bar{\Omega}[1+P] - \frac{\bar{\varepsilon}\bar{x}^2}{\eta})\xi_o + (\bar{\varepsilon} \ \frac{\bar{x}}{\eta} - P\bar{\Omega})\psi_o$$

In the first order:

$$\eta \frac{d^2 \psi_1}{d \bar{x}^2} = -\bar{x}\xi_o + \bar{\varepsilon}\bar{x}\zeta_o + \psi_o \tag{31}$$

Thus

$$\psi_o = B_{II}^{(0)} + A_{II}^{(0)} \bar{x}$$

But since $A_{II} \simeq 0$ for matching, we are justified in setting $A_{II}^{(0)} = 0$ in the zeroth order and accounting for the x dependence in the next order. In fact, from (31), and in view of (14) and (15), we have

$$\frac{d^2 \psi_1}{d\bar{x}^2} \simeq 0 \quad \text{as} \quad x \to \pm\infty$$

which has solutions,

$$\psi_1 \simeq \begin{cases} A_{II,L}^{(1)} \bar{x} \text{ as } \bar{x} \to -\infty \\ \\ A_{II,R}^{(1)} \bar{x} \text{ as } \bar{x} \to \infty \end{cases}$$

where the constant solution has been incorporated in the zeroth order solution. Thus the total ψ solution is,

$$\psi(\bar{x}) \simeq s^{7/4} A_{II}^{(I)} \bar{x} + s^{1/4} B_{II}^{(0)}$$

And since,

$$\frac{d\psi}{dx}(x) = \frac{1}{s^{1/4}} \frac{d\psi}{d\bar{x}}(\bar{x})$$

making use of (25), (26) and (31), we arrive at the dispersion relation,

$$\frac{1}{s}\Delta = \Delta' \tag{32}$$

where,

$$\Delta' = \frac{A_{III}}{B_{III}} - \frac{A_I}{B_I} \tag{33}$$

and,

$$\frac{1}{\phi} \Delta = \lim_{X \to \infty} \frac{\frac{d\psi}{dx}(X) - \frac{d\psi}{dx}(X-)}{\phi \beta S^{1/4}} = \frac{S^{5/4}}{\beta \phi} \lim_{X \to \infty} \int_{-X}^{-X} [-\bar{x}\xi_o + \bar{\epsilon} \bar{x} \gamma_o + \beta] d\bar{x} \tag{34}$$

where we have set $\beta \equiv B_{II}^{(0)} \equiv \psi_o$.

Thus the condition that $\Delta' = 0(1)$ requires that

$$\phi = S^{5/4} \tag{35}$$

9. The Tearing Mode Equations:

From the original scalings of section 5, and equation (34), the tearing mode scalings can be rewritten in terms of the scale factor θ as

$$\left.\begin{array}{l} \eta = \theta^5 \bar{\eta} \; , \; \gamma = \theta^3 Q \; , \; \dfrac{d\Omega}{dr} = \theta \dfrac{d\Lambda}{dr} \\[2mm] \tau = \theta^2 \bar{x} \; , \; \varepsilon = \theta \bar{\varepsilon} \; , \; \Omega = \theta \Lambda \end{array}\right\}$$

$$q_r = \theta^2 \psi_o + \theta^4 \psi_1 + \ldots$$

$$\xi_r = \xi + \ldots \tag{36}$$

$$q_\theta = \theta^2 \zeta_o + \ldots$$

$$\psi_o = \beta$$

and letting $P \approx \dfrac{r_o}{2\Lambda} \dfrac{d\Lambda}{dr}$ and $q = \dfrac{\varepsilon D_S}{\Lambda}$. $\tag{36a}$

The following tearing mode equations result from scaling the basic equations (11) - (13) according to (36), omitting superscripts and considering $\left| P - \dfrac{r_o D_S}{4} \right| \ll 1$

$$\frac{d^2 \xi_o}{dx^2} = \left(\frac{\Lambda^2}{Q^2} \left[1 - \frac{r_o D_S}{4} + P \right] + \frac{x^2}{Q\eta} \right) \xi_o + \left(\frac{\Lambda x}{Q^3} - \frac{\varepsilon x^2}{Q^2 \eta} \right) \xi_o$$

$$+ \left(\frac{\alpha_1 - \frac{1}{r_o}}{Q^2} - \frac{x}{\eta Q} \right) \beta \tag{37}$$

$$\frac{d^2\zeta_o}{dx^2} = (\frac{x\Lambda}{\eta}[1 + p+q] - \frac{\varepsilon x^2}{\eta^2} Q)\xi_o + (\frac{x^2}{Q} + \frac{\varepsilon^2 x^2}{\eta^2})\zeta_o$$

$$+ (- \frac{\varepsilon}{\eta}[\alpha_1 - \frac{1}{r_o}] - \frac{\Lambda}{\eta}[p+q] + \frac{\varepsilon x Q}{\eta^2})\beta \tag{38}$$

$$\frac{d^2\psi_1}{dx^2} = - \frac{xQ}{\eta}\xi_o + \frac{\varepsilon x}{\eta}\zeta_o + \frac{Q}{\eta}\beta \tag{39}$$

where the superscripts have been omitted from the scaled variables.

We study the asymptotics of system (37) - (39) by expanding the independent variables as follows:

$$\xi = \frac{\xi_1}{x} + \frac{\xi_2}{x^2} + \ldots \tag{40}$$

$$\zeta = \frac{\zeta_1}{x} + \frac{\zeta_2}{x} + \ldots \tag{41}$$

Replacing these expansions in (37) - (39), we obtain the result that

$$\left.\begin{array}{l} \zeta_1 = 0 \quad, \quad \xi_1 = \beta \ , \\[2mm] \zeta_2 = -\beta Q\Lambda \ , \quad \xi_2 = [\frac{\eta}{Q}(\frac{1}{r_o} - \alpha_1) - \varepsilon\Lambda]\beta \end{array}\right\} \tag{42}$$

and,

$$\psi_1 = \beta[1 + \frac{\Lambda^2 r_o D_s}{4} \ln x + \ldots] + [c_1 + \beta(\alpha_1 - \frac{1}{r_o}) \ln x]x + \ldots$$

where β and c_1 are chosen to match the outer solution.

10. Analysis of the Tearing Mode Equations:

In order to study system (37) - (39) analytically we are motivated by equations (42) to make the following change of variables:

$$z = \frac{x}{(Q\eta)^{1/4}}$$

$$\zeta = \frac{Q^{1/2}\Lambda\beta}{\eta^{1/2}} \bar{\zeta}$$

$$\xi = \frac{\beta}{(Q\eta)^{1/4}} \bar{\xi}$$

We also define the following parameters,

$$r^2 = \frac{\epsilon^2 Q}{\eta} \quad , \quad \Lambda = \frac{Q^{3/4}}{\eta^{1/4}} \bar{\Lambda}$$

$$\text{and } \bar{q} = P + q$$

$$\tag{43}$$

With these substitutions (37) - (39) can be rewritten, assuming that $\left|\frac{r_o D_S}{4} - P\right| \ll 1$:

$$\frac{d^2\bar{\xi}}{dz^2} = [\bar{\Lambda}^2(1 - \frac{r_o D_S}{4}) + z^2]\bar{\xi} + \bar{\Lambda}[\bar{\Lambda}z - rz^2]\bar{\zeta} + (\alpha_1$$

$$- \frac{1}{r_o}) \frac{\eta^{3/4}}{Q^{5/4}} - z \tag{44}$$

$$\frac{d^2\zeta}{dz^2} = \frac{1}{\bar{\Lambda}} (z\bar{\Lambda}(1 + \bar{q}) - rz^2)\bar{\xi} + z^2(1 + r^2)\bar{\zeta}$$

$$+ (\frac{rz}{\bar{\Lambda}} - \frac{\varepsilon\eta^{1/4}}{Q^{3/4}\bar{\Lambda}} (\alpha_1 - \frac{1}{r_o}) - \bar{q} \tag{45}$$

$$\frac{d^2\bar{\psi}}{dz^2} = \frac{Q^{3/2}}{\eta^{1/2}} (1 - z\bar{\xi} + r\bar{\Lambda}\bar{\zeta}) \tag{46}$$

In the text that follows we will simplify the computation by taking $\alpha_1 = \frac{1}{r_o}$.

The dispersion relation (32) - (34) can also be rewritten in terms of these substitutions:

$$\frac{1}{\beta} \frac{d\psi}{dx} = \frac{Q^{5/4}}{\eta^{3/4}} F$$

where,

$$F = \lim_{z \to \infty} \int_{-z}^{z} (1 - z\bar{\xi} + r\bar{\Lambda}\bar{\zeta})dz \tag{47}$$

and,

$$\frac{1}{\beta} \frac{d\psi}{dx} = \frac{1}{\beta} \lim_{X \to \infty} [\frac{d\psi}{dx} (X) - \frac{d\psi}{dx} (-X)] = \Delta.$$

Thus, the dispersion relation is:

$$\Delta' = \Delta \tag{48}$$

and making use of (36) we have:

$$\bar{\gamma} = \bar{\eta}^{3/5} \left(\frac{\Delta'}{F}\right)^{4/5} \tag{49}$$

where the dimensionless quantities $\bar{\gamma}$ and $\bar{\eta}$ have been used in place of the scaled variables Q and η , and F depends on the parameters in equations (44) - (46), namely, $\bar{q}, r, \bar{\Lambda}$, and D_s.

A Special Case:

Our objective in this section is to find a special class of analytic solutions to the inner layer equations (44) and (45), and use these solutions to evaluate the dispersion relation (48) explicitly.

The case we consider is characterized by,

$$\frac{r_o D_s}{4} \ll 1, \quad r \ll 1, \quad P = 0, \quad \bar{q} = -1, \quad \bar{\Lambda} = 0(1)$$

This corresponds to a Theta pinch equilibrium with rigid body rotation and density gradient opposite in sign to the angular velocity.

From (44) we see that only the odd part of $\bar{\xi}$
and $\bar{\zeta}$ contribute to the dispersion relation. Furthermore,
in the limit of small Q, the second term in (44) can
be treated as a perturbation, thus only the odd part
of $\bar{\xi}$ must be determined in order to derive the dispersion
relation. But, referring to (43) and (44), and using
the result of the asymptotics summarized in (14) and (15),
all terms involving r may be treated as pertubations and
neglected to leading order. Thus (44) - (46) reduces to:

$$\frac{d^2\bar{\xi}}{dz^2} - z^2\bar{\xi} - \bar{\Lambda}_0^2 \bar{\xi} = z \bar{\Lambda}_0^2 \bar{\zeta} - z \tag{50}$$

$$\frac{d^2\bar{\gamma}}{dz^2} - z^2 \bar{\zeta} = 1 \tag{51}$$

$$\frac{1}{\beta} [\frac{d\psi}{dx}] = \frac{1}{(\eta Q)^{1/4}} \int_{-\infty}^{\infty} \frac{Q^{3/2}}{\eta^{1/2}} (1 - z\bar{\xi}) \ dz \tag{52}$$

where we denote the value of $\bar{\Lambda}$ corresponding to the
special case q = -1 by $\bar{\Lambda}_0$. We let,

$$\bar{\zeta} = \epsilon^{-z^2/2} \sum b_{2k} H_{2k}(z) \tag{53}$$

$$\bar{\xi} = \epsilon^{-z^2/2} \sum a_{2k+1} H_{2k+1}(z) \ , \tag{54}$$

and make use of the identities,

$$
\left.
\begin{aligned}
1 &= \sqrt{2}\; \varepsilon^{-z^2/2}\; \frac{H_{2k}(z)}{2^{2k}\Gamma(k+1)} \\[2em]
z &= \sqrt{2}\; \varepsilon^{-z^2/2}\; \frac{H_{2k+1}(z)}{2^{2k}\Gamma(k+1)}
\end{aligned}
\right\}
$$

Solving for $\bar{\xi}$ we find,

$$
a_{2k+1} = \frac{1}{2^{2k+\frac{1}{2}}\Gamma(1+1)} \; \frac{[2 + \bar{\Lambda}_o(\frac{1}{4k+1} + \frac{1}{4k+5})]}{(4k+3 + \bar{\Lambda}_o^2}
$$

Replacing $\bar{\xi}$ in (51), and using the orthogonality relations for Hermite polynomials yields,

$$
\Delta = \frac{\pi \; \bar{\gamma}^{-5/4}}{2 \bar{\eta}^{-3/4}} \; F(\bar{\Lambda}_o)
$$

where,

$$
F(\bar{\Lambda}_o) = -4(1 + \bar{\Lambda}_o^2) \; \frac{\Gamma(\frac{3+\bar{\Lambda}_o^2}{4})}{\Gamma(\frac{5+\bar{\Lambda}_o^2}{4})} + 2\bar{\Lambda}_o^2 \; \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \quad .
$$

Growth rates can be obtained by solving the dispersion relation, $\Delta' = \Delta$, for the growth rate as given by (49) where the quantity Δ' is determined by integrating the ideal equations of motion forward from $r = 0$ and backward from $r = r_w (r_w$ = wall position) and then forming the logarithmic derivative as indicated in Eq. (30a). From (49) we notice

that the stability criterion is $\Delta' > 0$, that is, for positive values of Δ' the plasma is unstable and positive growth rates can be calculated. For $\Delta' \leq 0$, which depends only on equilibrium quantities outside the resistive layer, the plasma is stable. We note that, in general, the dispersion relation will have to be solved iteratively since from (43) , $\bar{\Lambda}_o$ depends on the growth rate.

11. Validation of the Analysis

The ultimate goal in this work is to compute the stability (growth rates and stability boundaries) of realistic linear plasma configurations to the tearing modes considered here. Before proceeding, it is improtant to validate the analysis with a specific example. Let us consider the special case of the last section. We will show by a numerical integration of the governing equations in the resistive layer that i) the asymptotics are correct and ii) the analytic solutions given by (53) and (54) are correct. We use a technique for solving Eqs. (50) - (52) which converts the boundary value problem into an initial value problem [12]. This technique has been used successfully elsewhere in a number of resistive boundary layer calculations [13]. We have chosen the special case $q = -1, \bar{\Lambda}^2 = 0.75$ and have numerically integrated the resistive layer equations for $\bar{\xi}$ and $\bar{\gamma}$ as a function of z

for $0 \leq z \leq 4$. Shown in figure 2 is a curve of $\bar{\xi}$ vs. z; the crosses denote the solution as given by the analytic expression (89), and the circles denote the results from the numerical integration of Equation (50) and (51). Figure 2 presents a similar comparison between analytic and numerical solutions for $\bar{\gamma}$ vs. z. The asymptotic expansions for $\bar{\xi}$ and $\bar{\gamma}$ for large z, i.e. $\bar{\xi} \sim 1/z$ and $\bar{\gamma} \sim 1/z^2$, are borne out by both the numerical and analytical solutions. Further, comparison of the solutions for the entire z range indicates excellent agreement. We therefore believe that the numerical technique for solving the RLE equations is satisfactory.

12.   Effect of Hall current on the growth rate:

From equation (49) we see that the effect of Hall
current on the growth rate is entirely contained within
the quantity $F(\bar{\Lambda},q,r,D_s)$. The value of F is governed,
for the most part, by the parameters of the inner layer,
namely, ϵ,η and the local values of the angular velocity,
Ω, and the density gradient, D_s. Thus the quantity F
maybe considered as a criteria of merit which gives us
a means of evaluating the effect of Hall current on the
growth rate of the tearing mode as compared to growth
rates in the absence of Hall current. For example in
the absence of Hall current and the case of rigid body
rotation, $\epsilon = \bar{q} = 0$ and $F = F(\bar{\Lambda},0,0,D_s)$. Thus larger
values of F in the presence of Hall current compared to
no Hall current would according to (49) tend to depress
the value of the growth rate whereas smaller values of F
would tend to have the reverse effect of enhancing the
growth of the tearing mode.

It is important, therefore, to examine the criteria
of merit, F, more closely. We will demonstrate later that
for realistic values of the parameters, F depends only on
$\bar{\Lambda}^2$ and $\bar{q}$. In other words $F = F(\bar{\Lambda}^2,\bar{q})$, and the parameter r
along with the explicit dependence of F on D_s through the
term $\frac{r_o D_s}{4}$ in equation (44) may be considered as small pertur-
bations. This simplifies the analysis of the inner layer con-
siderably and enables us to illustrate the dependence of

F on $\bar{\Lambda}^2$ and $\bar{q}$ by Figure 3. If we consider rigid body rotations, the curve $\bar{q} = 0$ corresponds to the case of no Hall current. From this figure it is evident that values of $\bar{q} > 0$ result in enhanced stability to tearing modes whereas values of $\bar{q} < 0$ result greater instability. For the case of rigid body rotation, from (43), $\bar{q} = \dfrac{\varepsilon D_s}{\bar{\Omega}}$. Thus rotation of the plasma in one direction relative to the value of the density gradient at the neutral point, enhances the stability to tearing modes while rotation in the opposite direction results in greater instability. This result was observed by the computer analysis of Kalek [7].

Figure 3 also shows that the effect of rotation on F, and therefore the growth rates, is comparitively small for the case of no Hall current due to the relative constancy of the curve $q = 0 = \varepsilon$ compared to curves with larger values of q.

It should also be noted that in the case of no Hall current but a sheared velocity profile, $\bar{q} = \dfrac{r_o}{2\bar{\Omega}} \dfrac{d\bar{\Omega}}{dr}$, and once again Figure 3 shows that sheared velocity can also effect the stability of tearing modes. However, this point will not be pursued in this paper. Hereafter, we will specialize our analysis to rigid body rotations.

13. <u>Effect of the equilibrium and position of the outer wall
on the tearing mode</u>:

The equilibrium determines the parameters of the inner
layer and thus influences the tearing modes through the
criteria of merit, F. The equilibrium also affects the
tearing modes through the outer equations via the parameter
Δ'. Furthermore, since by equation (49), the value of Δ'
is extremely sensitive to the position of the outer wall
surrounding the vacuum region, we see that growth rates and
thus values of the inner parameter, $\bar{\Lambda}^2$ defined by (43), are
very sensitive to the outer wall position. Thus values of
$\bar{\Lambda}^2$ determine not only the effect of rotation but also of the
position of the wall on the value of the criteria of merit,
F.

We now show how the Hall current rigid body rotation,
and wall position effect the growth rates of tearing modes
for a family of rigid rotor, Theta pinch equilibria. To be
exact, we will determine an overestimate of the ratio of
growth rate with Hall current to growth rate with no Hall
current, and show how these ratios are affected by angular
velocity for fixed positions of the outer wall. The next
section is devoted to describing the equilibria.

14. A family of equilibria:

In this section we apply the theory to a class of reversed field, isothermal, rigid-rotor Theta pinch MHD equilibria. We assume that all the mass and rotational energy is carried by the ions and that the family of equilibria is parametized by the angular velocity, Ω, and characterized by a constant number of particles, N_L. The equilibria are one dimensional, and although their domain of definition is from the plasma center line, $r = 0$, to $r = \infty$, the density is effectively zero for $r \geq a$; where a is taken to be the plasma radius. Furthermore all equilibrium quantities are constant from the plasma edge to the wall of a vacuum region, r_w, surrounding the plasma, and the plasma column has length L and is topologically homologous to a torus. Figure 1 illustrates this configuration.

The equilibrium is characterized by the force balance,

$$\frac{d}{dr} \left(P + \frac{1}{2\mu_o} B^2 \right) = \Omega^2 r\rho \tag{55}$$

and the equation of state,

$$P = c\rho \quad , \quad c = \frac{KT}{m_i} \tag{56}$$

where K is Boltzmann's constant, m_i is the mass of an ion, and T is the plasma ion temperature.

The number of particles (ions), N_L, is determined by,

$$N_L = \frac{2\pi}{m_i} \int_o^a \rho(r) r dr$$

and expressed in terms of a characteristic length, r_ℓ, as

$$N_L = \frac{\pi}{m_i} \rho(o) r_\ell^2 \tag{57}$$

We consider the following ansatz,

$$\rho(x) = \rho(o) \operatorname{sech}^2 \left(\left(\frac{x}{x_p} \right)^2 - \xi_o \right) \tag{58}$$

and,

$$B_z = B_\infty \left(\delta + (1-\delta) \tanh \left(\left(\frac{x}{x_p} \right)^2 - \xi_o \right) \right) \tag{59}$$

where, all lengths are normalized to the plasma radius,
i.e., $x = \frac{r}{a}$.

Replacing (58) and (59) in (55) and (56) and characterizing the ratio of the thermal to the magnetic energy by,

$$\beta = 2c \frac{\rho(o) \mu_o}{B_\infty^2} ,$$

and the ratio of the rotational to the thermal energy by

$$\Omega^{*2} = \frac{\Omega^2 a^2}{2c} \beta$$

yields,

$$\Omega^{*2} = \frac{2\delta(1-\delta)}{x_p^2} \text{ sech}^2 \xi_o$$

$$\beta = (1-\delta)^2 \text{ sech}^2 \xi_o$$

$$x_p^2 = x_\ell^2 \frac{\text{sech}^2 \xi_o}{1+\tanh \xi_o}$$

The equilibrium is therefore entirely specified once the physically meaningful variables, B_∞, $\rho(o)$, T, N_L and Ω are specified. However, for our purposes it is more convenient to specify, B_∞, N_L, ξ_o, δ, and x_ℓ.

In terms of the latter set of parameters, the plasma temperature, rigid body rotion and the neutral point, x_o, of the plasma are given by,

$$\Omega^2 = \frac{2\pi\delta(1-\delta)(1+\tanh \xi_o)}{m_i \mu_o N_L} \tag{60}$$

$$T = \frac{m_i c}{K}$$

where,

$$c = \Omega^2 a^2 x_\ell^2 (1-\tanh \xi_o) \left(\frac{1-\delta}{4\delta}\right)$$

and,

$$x_o = x_\ell \ (1-\tanh \xi_o)^{1/2} \ (\xi_o + \tfrac{1}{2} \ln(1-2\gamma))^{1/2}$$

The object of this study is to show the effect of the Hall current dominated boundary layer on the growth rates, γ, of the tearing modes expressed by the dispersion relation,

$$\bar{\gamma} = \bar{\eta}^{3/5} \left[\frac{\Delta'}{F(\bar{\Lambda}^2,q)} \right]^{4/5} \tag{61}$$

where, once again, the superscripted variables are normalized with respect to the boundary layer quantities and the variables q and $\bar{\Lambda}^2$ governing the inner equations are defined by (36a), and (43). Values of the mismatch, Δ', are obtained by integrating the outer equations as described in section 7 while $F(\bar{\Lambda}^2, q)$ is obtained by integrating the inner equations as described in section 8. . Since $\bar{\gamma}$ occurs on both sides of the equation, (61) must be solved iteratively.

The inner variables and the normalizations for (61) are computed as follows:

$$q = \frac{\varepsilon\Omega}{1-\delta} \ (\xi_o + \tfrac{1}{2} \ln(1-2\gamma)) \tag{62}$$

$$\bar{\Lambda}^2 = \frac{\bar{\eta}^{-1/2} \bar{\Omega}^2}{\bar{\gamma}^{3/2}} \tag{63}$$

$$r^2 = \frac{2\epsilon^2 \, \delta \, c\mu_o}{\eta(1-2\delta)}$$

$$\bar{\eta} = \frac{\eta}{a\bar{k}\mu_o V}$$

$$\bar{\Omega} = \frac{2\Omega a}{V}$$

$$\bar{\gamma} = \frac{\gamma a}{kV}$$

(64)

where,

$$\bar{k} = 2\pi \frac{a}{L}$$

and,

$$V^2 = \frac{4a^2\pi\left(\xi_o + \frac{1}{2}\ln(1-2\delta)\right)(1+\tanh\xi_o)(1-2\delta)}{\mu_o \, m_i \, N_L}$$

and from equation (21a)

$$h = -\frac{1}{2} + \frac{1}{2}\left[1 + \frac{8\delta^2}{1-2\delta}\right]^{1/2}$$

(65)

Results of tearing mode analysis:

We carried out the analysis of the above equilibrium
for the following range of parameters:

$$B_\infty = 1 \ , \ \xi_O = 1 \ , \ x_\ell = 1 \ ,$$

$$\epsilon = 1.02 \times 10^{-8} \ \epsilon - \frac{MHO\text{-}KG}{m^2 \ weber} \ , \ \eta = .12 \ \frac{mho\text{-}kg}{m^4}, \ \mu_O = 4\pi \times 10^{-7}$$

$$K = 1.38 \times 10^{-22} \ \frac{kg - m^2}{^o_k \ sec^2} \ , \ m_i = 3.2 \times 10^{-27} kg$$

$$a = .05 \ m \ , \quad \frac{a}{L} = .05$$

$$N_L = 1 \times 10^{16} \ ions \ to \ N_L = 1 \times 10^{18} \ ions$$

and,

$$\delta = .025 \quad to \quad \delta = .425$$

where δ is the normalized angular velocity defined by (60).

As we discussed in the previous sections, the effect
of the resistive layer on the tearing modes is contained
entirely in the criteria of merit, F, whereas the outer
regions affect the tearing modes primarily through the
mismatch Δ'. Finally, there is a feedback between outer
and inner regions that is particularly sensitive to the
position of the outer wall, since the position of the
outer wall directly affects Δ' and indirectly affects F
through $\bar{\Lambda}^2$. As we mentioned above, F depends to leading
order, on the parameters q and $\bar{\Lambda}^2$, since for the equilibria

being considered, the parameter $r < 10^{-3}$ while $\frac{r_0 D_s}{4}$, according to Table 1, varies from 0 to-.23. This range of values of $\frac{r_0 D_s}{4}$ has been shown to have only a minor effect on the results.

More important, as shown in section 8, it is basic to the tearing mode analysis that the exponent, h, defined by equation (21a), be real. Furthermore, the asymptotic analysis requires that h $\overset{\sim}{\sim}$ 0. Equation (65) shows that real h is guaranteed for $\delta < .5$, while Table 1 inicates that reasonable results can be expected from the asymptotic analysis for $\delta < .2$. However, we carry out the analysis for values of δ as high as .425 and expect the results to have qualitative validity.

Exact values of the growth rate could be carried out using the following procedure:

1. Fix the position of the outer wall at some point at which the equilibrium has an unstable tearing mode for the above values of the equilibrium parameters and angular velocity corresponding to $\delta = .025$. As mentioned above, the criteria for such an instability is that $\Delta' > 0$ where Δ' is computed from equation (30b).

2. At this low value of the angular velocity, the parameter $\bar{\Lambda}^2$ governing the inner equations is effectively zero. Thus the criterion of merit, F, that characterizes the effect of Hall current and rotation, takes a value approximately equal to its vertex value of $F = 2.12$ as shown in Figure 3. The normalized growth rate, $\bar{\gamma}$, can then be calculated from:

$$\bar{\gamma} = \bar{\eta}^{3/5} \left(\frac{\Delta'}{F}\right)^{4/5}$$

where $\bar{\eta}$ is computed from (64).

3. The angular velocity specified by δ is increased by a small amount keeping the particle number, N_L, fixed. The outer equations are reintegrated to obtain a new value of Δ'. The value of q is determined from equation (62) and since F depends also on $\bar{\Lambda}^2$ which by equation is a function of the growth rate $\bar{\gamma}$, the growth rate must be determined this time from equation (61) by iteration. This iteration is initiated by using the value obtained for $\bar{\gamma}$ from step 2 as the first guess.

4.   Step 3 is repeated for the case of no Hall current, i.e., $\epsilon = q = 0$. The values of the growth rates, are used as normalizing factors; i.e., γ normalized $= \dfrac{\bar{\gamma}}{\gamma_{\epsilon} = 0}$.

In this way a table of normalized growth rates as a function of angular velocity for a constant wall position would be obtained for one family of equilibria parametized by N_L. However there is one problem which makes the implementation of the above procedure difficult. The determination of Δ' requires the evaluation of (30b) at the edge of the boundary layer. However, as shown in Table 1, the thickness of the boundary layer, $\tau_{b.\ell.}$, as calculated by equation (23), varies directly with angular velocity taking on values on the order from 0 to 10^{-3}. However, this close to the neutral point, where $B_z(r_o) = 0$, the solutions of the outer equations are asymptotically large and the integration routines are no longer sufficiently accurate. It turns out that for some parameter ranges there exists a small annulus about the neutral point in which the plasma position is close enough to the neutral point for a computation of Δ' to be made yet for enough away from the neutral point to enable the outer equations to be accurately integrated, but this is generally not the case.

We can avoid this difficulty of integrating the outer equations if we are willing to settle for overestimates of the growth rates and incomplete information as to the

exact position of the outer wall. We merely fix the value
of Δ' at its maximum value over the range of values on the
interval of angular velocities of interest. This corre-
sponds to some position of the outer wall. We see from
equation (61) that this value of Δ' yields an overestimate
of $\bar{\gamma}$ whereas from equation (63) for $\bar{\Lambda}^2$, overestimates of
$\bar{\gamma}$ yield smaller values of $\bar{\Lambda}^2$ for the same value of q thus
from Figure 3 this results in lower values of F and once
again from equation (61), overestimates of $\bar{\gamma}$. We have
implemented this procedure and presented the results in
Figures 4-6, where $\frac{\gamma}{\gamma_{\epsilon = 0}}$ is plotted against the normalized
angular velocity, δ, for different but unknown, outer wall
positions characterized by and varying directly with Δ'
and for three values of the number of particles,
$N_L = 10^{16}$, 10^{17}, 10^{18}. Actual values of the rigid body
angular velocity, Ω, and ion temperature T are indicated
in figure 4.

Figures 4-6 validate the previously stated effect,
namely, that the Hall current results in increased stability
to tearing modes when the plasma column rotates in one direc-
tion and decreased stability when the column rotates in the
opposite direction. Figures 4-6 show that, in general, both
stability and instability are enhanced by rotating the column
faster in either direction, although above a critical value
of the wall position, Δ', the normalized growth rate, $\frac{\gamma}{\gamma_{\epsilon = 0}}$,

attains a maximum with respect to the normalized angular velocity, δ, for rotations in one direction and a minimum for rotations in the opposite direction.

Figures 4-6 also show that the growth rates are very sensitive to the wall position, Δ', and that the plasma column becomes more stable to tearing modes for outer wall positions nearer to the plasma vacuum interface, i.e., for smaller values of Δ'. Furthermore, for sufficiently small values of Δ', at a critical value of the normalized angular velocity, δ, the tearing mode instability is quenched, i.e.,

$$\frac{\gamma}{\gamma_{\epsilon = 0}} = 0 .$$

Figures 4-6 also show that for a given wall position, Δ', the stabilizing effect is more pronounced at lower values of the number of particles, N_L, and higher ion temperatures, T.

Conclusions:

1.   Overestimates of the effect of Hall current and rotation on the growth rate of tearing modes can be determined from characteristics of the equilibrium at the neutral point of the reversed magnetic field provided that the equilibrium is known to be unstable to tearing modes.

2.   The effect of the tearing mode is characterized by the criteria of merit F which depends, to leading order, on only two parameters q and $\bar{\Lambda}^2$. When q > 0 the Hall current always enhances the stability of the plasma system to tearing modes while values of q < 0 always result in reduced stability. The parameter $\bar{\Lambda}^2$ is governed primarily by the position of the outer wall and the value of the angular velocity.

3.   For a given number of particles, the stabilizing effect of the Hall current is enhanced by wall positions closer to the plasma vacuum interface. For wall positions closer to the plasma than a critical value, the tearing mode instability is quenched at a critical value of the angular velocity of the column. For wall positions larger than this critical value, the growth rate attains a minimum at one value of the angular velocity for rotation in one direction, and attains a maximum for another value of angular velocity for rotation in the opposite direction.

4.   For a given wall position, the stabilizing effect of the Hall current is more pronounced for fewer numbers of particles, N_L, and higher ion temperatures, T.

Acknowledgements

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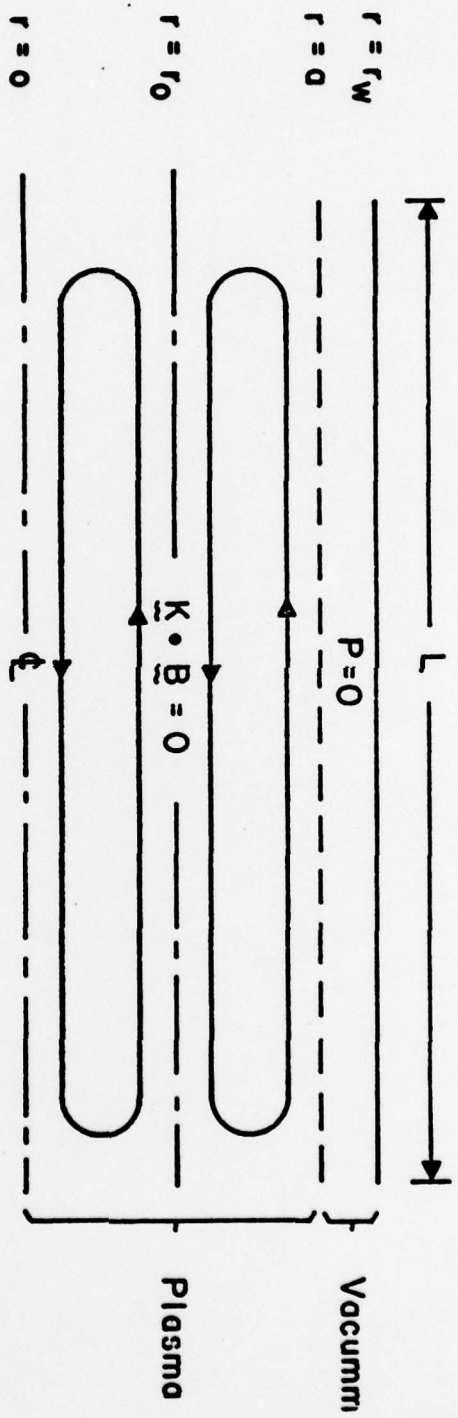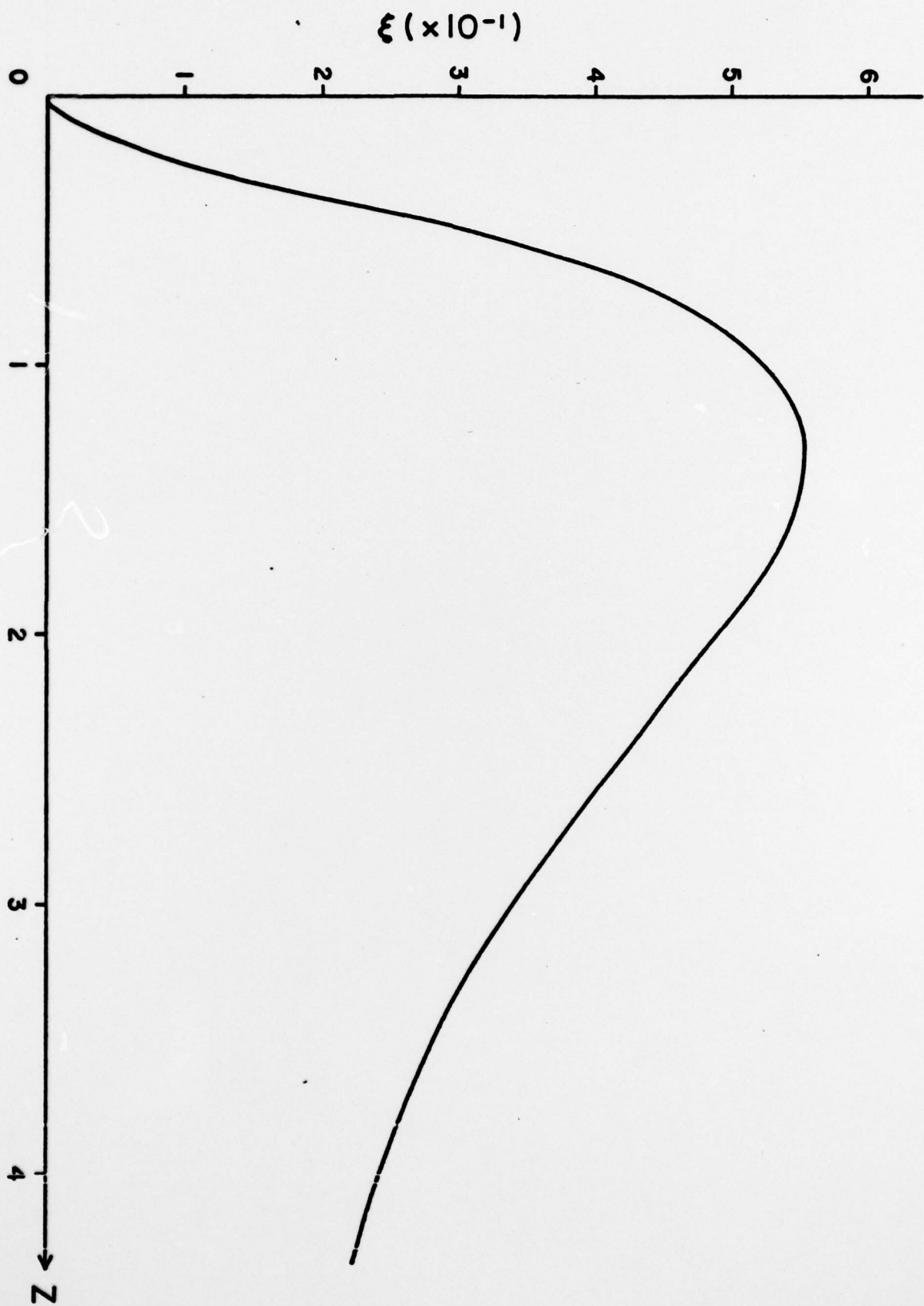| δ	X_o	$\dfrac{X_o D_s}{4}$	h	$\tau_{b.\ell.}$
0	.326	0	0	0
.1	.307	-.099	.024	3.8×10^{-5}
.2	.281	-.161	.119	1.87×10^{-4}
.3	.240	-.232	.337	5.4×10^{-4}
.4	.144	-.130	.860	1.15×10^{-3}

Table 1

Validity of the Analysis.

Figure Captions

Figure 1:   Configuration of the plasma system.

Figure 2a: Analytical solution, $\xi(Z)$, the resistive layer
            equations for a special case (numerical solution
            coincides with the analytic solution within the
            accuracy of the graph).

Figure 2b: Analytic solution, $\gamma(Z)$, of the inner equations for
            a special case (numerical solution coincides with
            analytic solution within the accuracy of the graph).

Figure 3:   Dependence of the criteria of merit, F, on the
            parameters $\bar{\Lambda}^2$ and q governing the resistive layer.

Figure 4:   Results of the analysis showing the effect of the
            normalized growth rate $\frac{\gamma}{\gamma_{\epsilon=0}}$ on the normalized angular
            velocity of the plasma δ, and a measure of the outer
            wall position, Δ', for particle number, $N_L = 10^{16}$.
            The actual angular velocity of the plasma column is
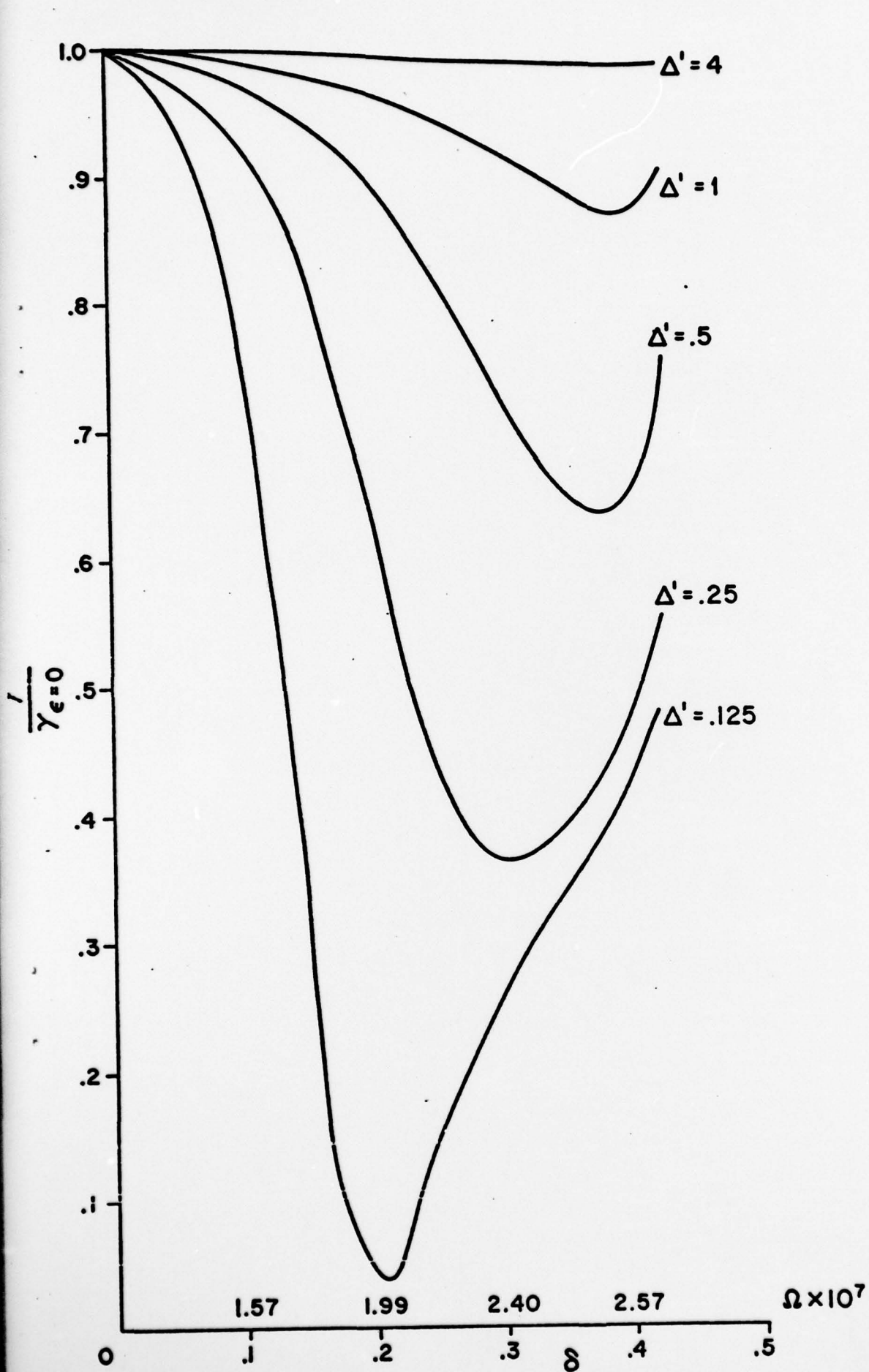   noted on the abscissa.

Figure 5:   Results of the analysis showing the effect of the
            normalized growth rate $\frac{\gamma}{\gamma_{\epsilon=0}}$ on the normalized angular
            velocity of the plasma δ, and a measure of the outer
            wall position, Δ', for particle number, $N_L = 10^{17}$.
            The actual angular velocity of the plasma column is
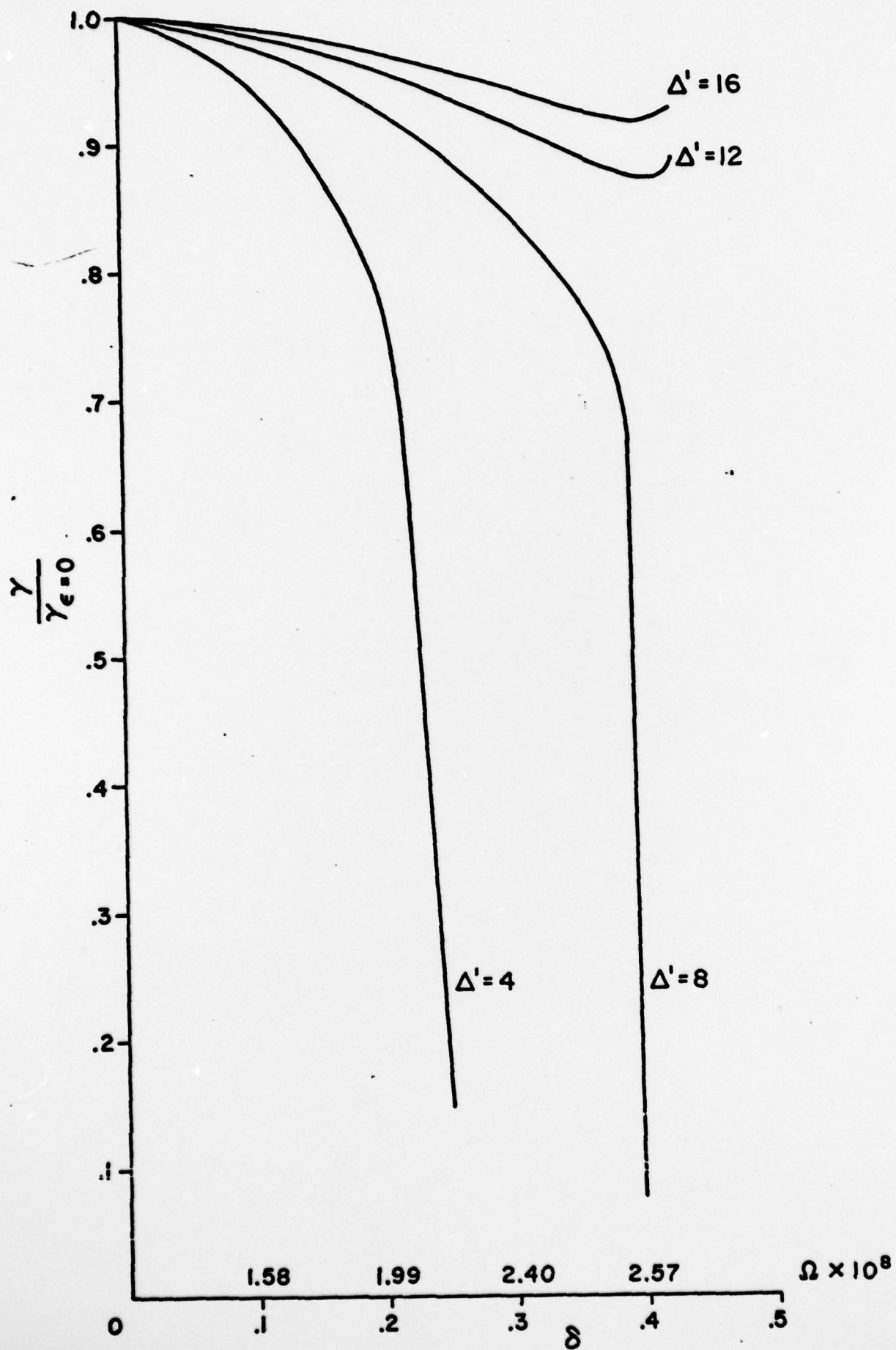   noted on the abscissa.

Figure 6: Results of the analysis showing the effect of the normalized growth rate $\frac{\gamma}{\gamma_{\varepsilon=0}}$ on the normalized angular velocity of the plasma δ, and a measure of the outer wall position, Δ', for particle number, $N_L = 10^{18}$. The actual angular velocity of the plasma column is noted on the abscissa.
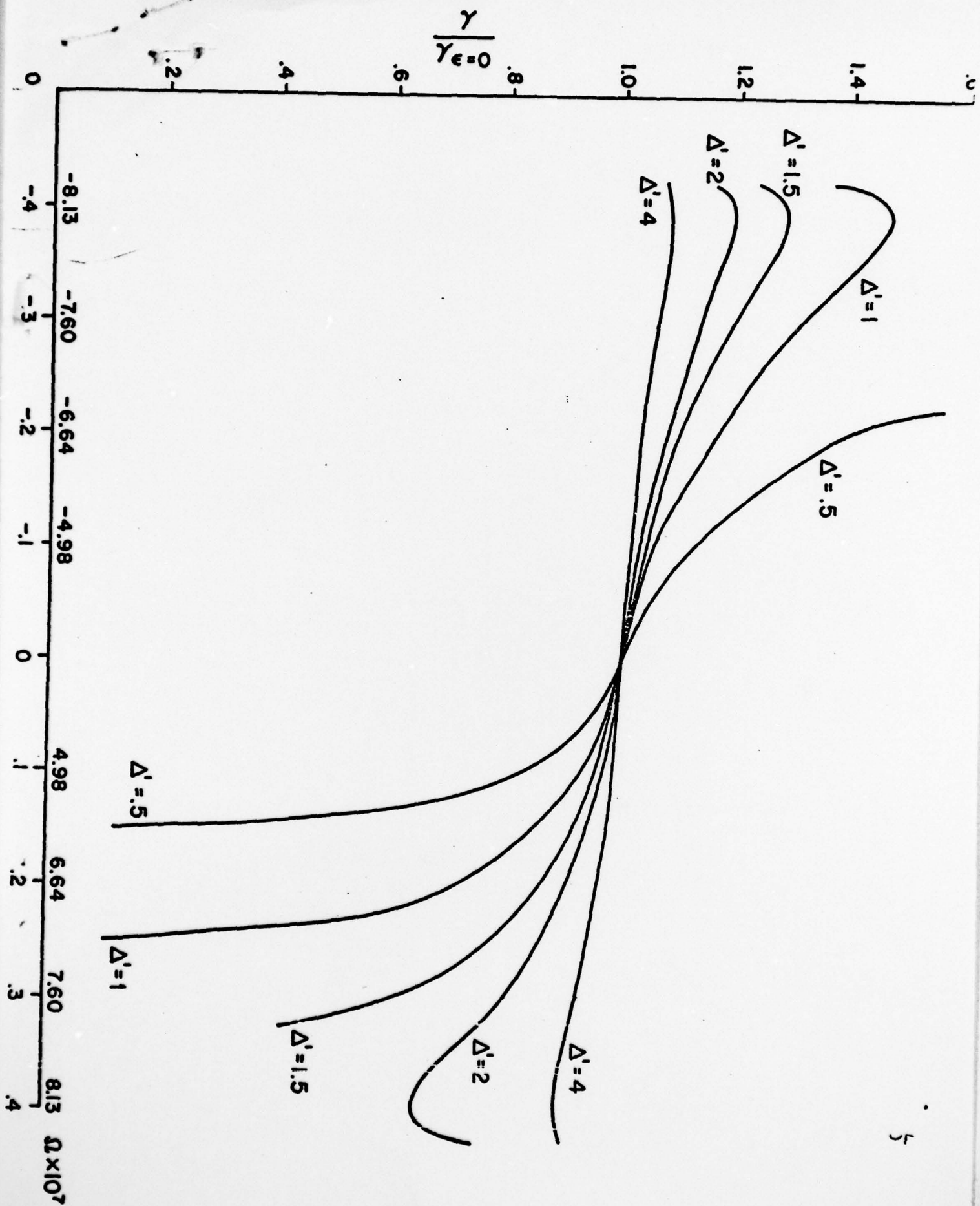
$r = r_w$

$r = a$

$r = r_o$

$r = 0$

L

$K \cdot B = 0$

$P = 0$

Plasma

Vacumm

$\xi \, (\times 10^{-1})$
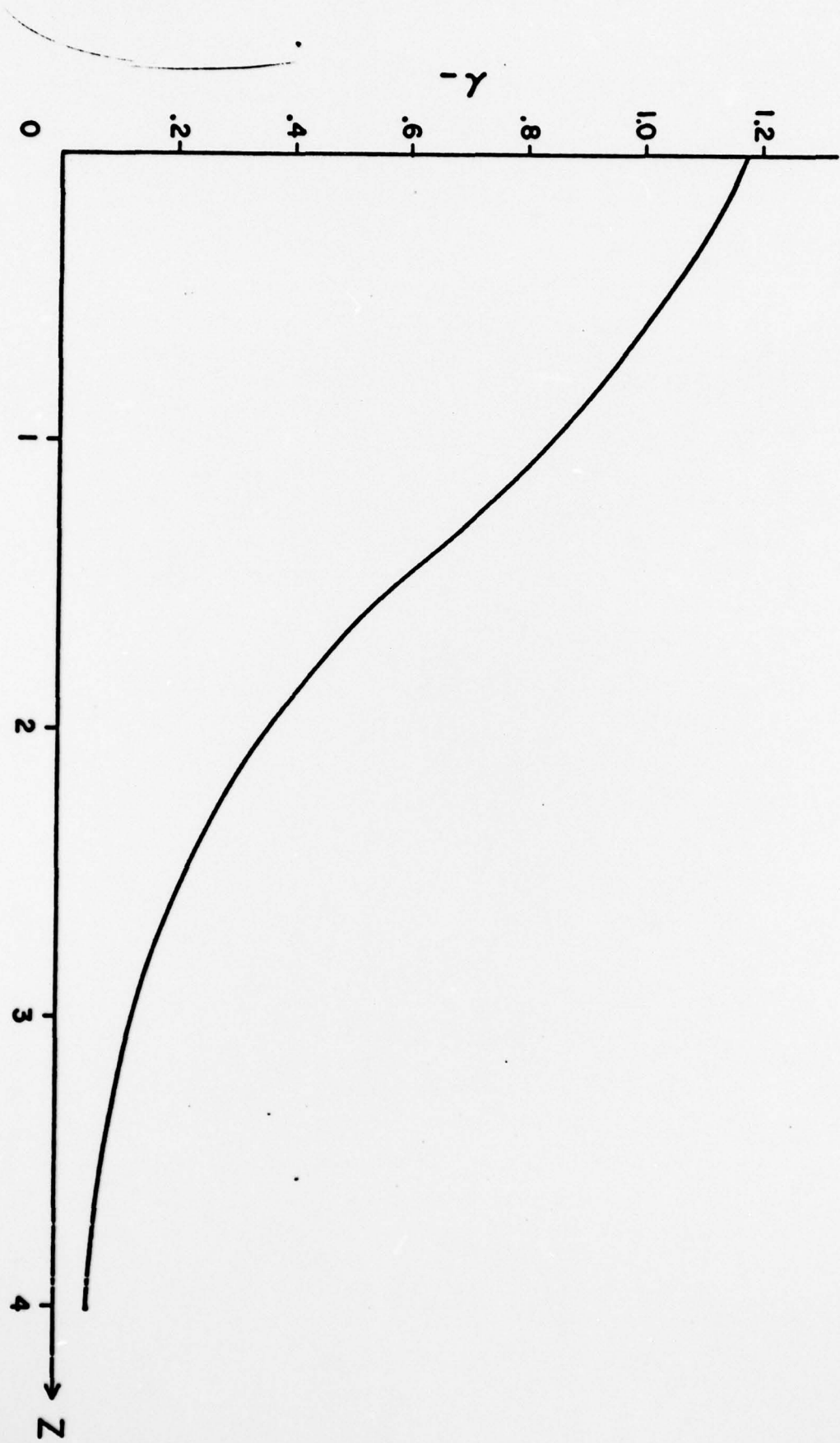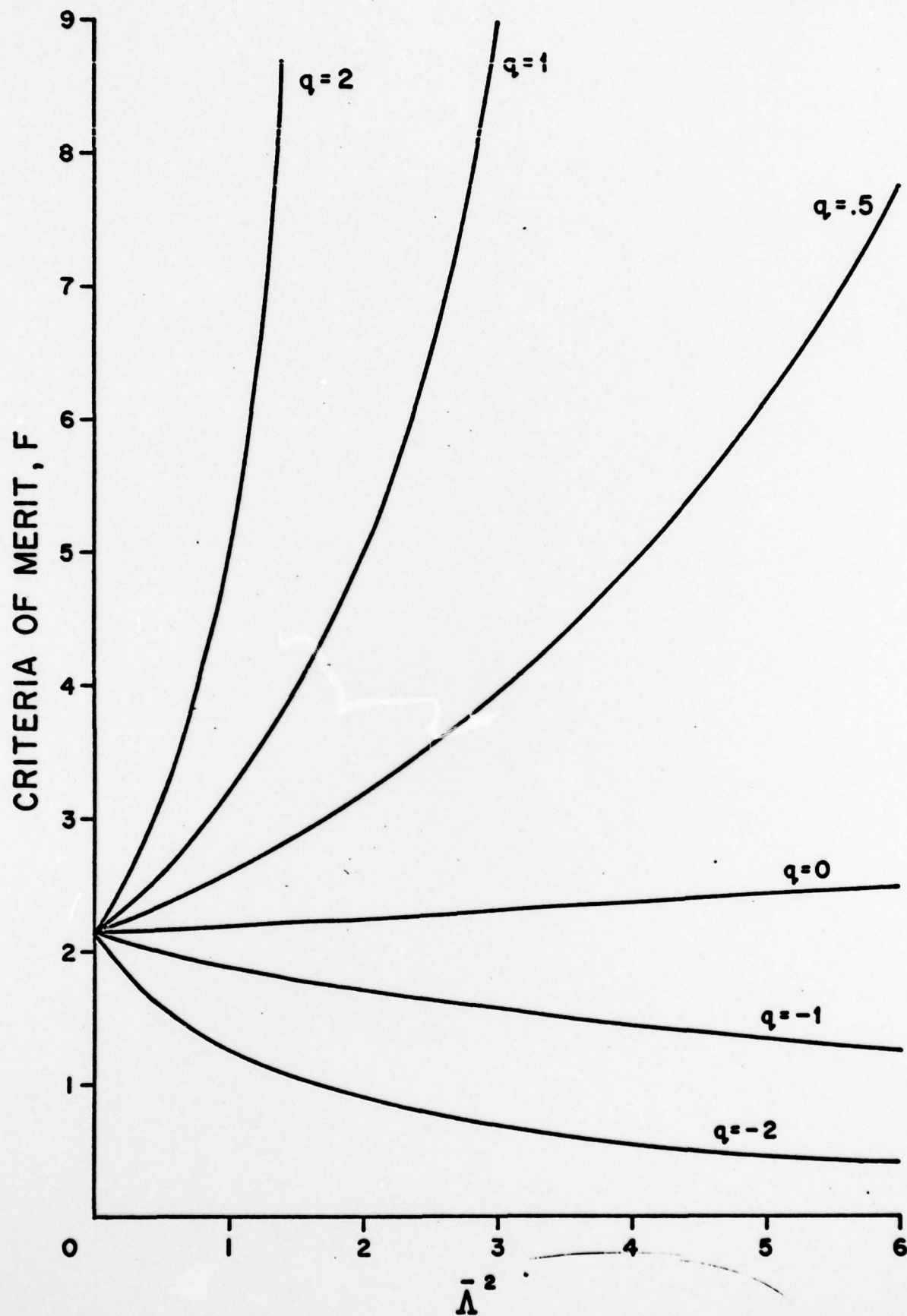
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The effect of rigid body rotation, Hall effect, and position of the outer retaining wall on the growth rates of tearing modes has been investigated for a family of reversed field Theta pinch equilibria. Paremeters sensitive to the physics of a boundary line surrounding the neutral surface of the plasma are identified and used to show how stability to tearing modes is enhanced by rotation of the plasma column in a preferred direction. Stabilization of the tearing modes is also found to be sensitive to well position and particle density and it is found that the tearing modes may → next page

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even be quenched at critical values of the angular velosity for outer
wall positions sufficiently close to the plasmar vacuum interface.